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THREE-DIMENSIONAL
THERMOELASTIC PROBLEMS OF PLANES
OF DISCONTINUITIES
OR CRACKS IN SOLIDS

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Abstract

Presented in this paper is a general formulation of the three-dimensional thermoelastic equations (uncoupled) for problems involving crack-like imperfections or planes of discontinuities of some kind in solids. Harmonic functions are constructed from which the stresses and displacements may be obtained. The thermoelastic potential for symmetric distribution of temperatures on the crack surfaces is related to Boussinesq's three-dimensional logarithmic potential for a disk in the shape of the crack. The mass density of the disk is found to be proportional to the prescribed temperature gradient normal to the crack plane. As an application of the theory, closed form solutions, in terms of complete and incomplete elliptic integrals of the first and second kind are given for a flat elliptical crack whose surfaces are exposed to uniform temperatures and/or temperature gradients. The possibility of extending the Griffith-Irwin theory of fracture to cracks in thermal environments is also discussed.

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Notation

a, b	Semi-axes of elliptical crack.
f, g, h	Harmonic functions of x, y, z.
k, k'	Arguments of complete elliptic integrals, $k^2 + k'^2 = 1$.
$k_j (j=1,2,3)$	Stress-intensity factors.
q(x,y)	Density distribution.
q_0, m_j, p_j, q_j (j=1,2,3)	Constants.
r	Distance normal to crack border.
s, t	Dummy variables.
u, v, w	Rectangular components of displacement.
u_0, v_0, w_0	Displacements on crack surface.
x, y, z	Rectangular coordinates.
A, B, C	Multiplying constants
E(k), K(k)	Complete elliptic integrals of second and first kind associated with k, respectively.
F, G, H	Potential functions of x, y, z.
I	Integral.
Q(s)	Function of s.
Q_0	Uniform temperature gradient.
R, R_0	Length.
T(x,y,z)	Temperature distribution.
T_0	Constant temperature.
Z	$(= 1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2})$
α	Coefficient of thermal expansion.
β	Angle subtended by r and x-axis.
ξ, η, ζ	Ellipsoidal coordinates.
$\lambda(s)$	Arbitrary function.

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μ	Shear modulus.
ν	Poisson's ratio.
ρ	Ratio of r to a.
$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$	Normal components of stress.
$\tau_{xy}, \tau_{yz}, \tau_{zx}$	Shear components of stress.
ϕ	Angle in parametric equation of ellipse.
$\underline{\phi}$	Vector potential.
ψ	Thermoelastic potentials.
ψ^*, ψ^{**}	Scalar potentials.
Λ, Ω	Functions in skew-symmetrical problems.

Introduction

While the solution of problems of thermal stress for thin plates and long cylinders has received considerable attention in the past[4], comparatively little work has been done on the problems of three-dimensional theory of thermoelasticity for solids containing imperfections. It is known that when a temperature field is disturbed by the presence of cracks or flaws, there is high elevation of the local temperature gradient accompanied by thermal stress. Thermal disturbances of this kind in some cases cause crack propagation resulting in serious damage of structural members. Hence, solutions to thermoelastic problems of cracks have immediate practical value in fracture mechanics.

The existing literature on the three-dimensional aspects of thermal stress around cracks has been limited to a few publications. Moreover,

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[4]An extensive reference of papers on thermoelasticity may be found in [1,2].

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previous works are confined to axially symmetric problems dealing with the "penny-shaped" crack. Such a restriction is mainly due to the mathematical difficulties of this class of problems. By the method of dual integral equations in the Hankel transforms, Olesiak and Sneddon [3] investigated the distribution of thermal stress in the neighborhood of a penny-shaped crack in an infinite medium. The prescribed temperatures on the upper surface of the crack are identical with those on the lower surface. Using the condition of symmetry with respect to the crack plane, they reduced the problem to one of mixed boundary conditions on the surface of a semi-infinite solid. The same problem was also treated by Deutsch [4]. The case when the thermal conditions are applied skew-symmetrically to the crack surfaces was considered by Florence and Goodier [5]. The present paper deals with a class of thermoelastic problems with cracks of more general shapes which include the circular cracks in [3-5] as special cases. It should be pointed out that the effects of both inertia and coupling between temperature and strain fields are neglected in the aforementioned works.

An effective way of solving three-dimensional boundary-value problems is to construct the general solution in terms of certain arbitrary functions dictated, in part, by the appropriate field equations and, in part, by the topology of the region of interest. These arbitrary functions are then determined such that the boundary conditions of the problem are fulfilled. This is precisely the approach adopted in the work to follow. With the aid of harmonic functions, a general solution of Navier's equation including thermal effects is obtained. By restricting the analysis to problems with cracks, stress and displacement expressions are derived for two types of problems, one of which concerns with stress systems which are symmetrical about the crack plane and the other with skew-

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symmetrical systems. In the former case, the steady-state temperature field for an arbitrary region of exposure (or region of the crack) may be determined from the Newtonian potential for a disk in the form of the crack plane. For the purpose of illustration, detailed solutions in ellipsoidal coordinates are obtained for the problem of heat applied to the surfaces of an "elliptically-shaped" crack.

Since the practical aspect of this paper is to establish a criterion of fracture for cracks in thermally stressed bodies, an examination of the stress field in the vicinity of the crack border is pertinent. The local stresses are found to have the same functional relationship and inverse square-root singularity as those obtained in isothermal elastic bodies subjected to surface tractions [6]. The significance of this result is that the Griffith-Irwin theory of fracture [7], originally developed for bodies maintained at constant temperatures, may now be extended to predict the onset of rapid crack propagation caused by thermal changes. More specifically, stress-intensity factors $k_j(j=1,2,3)$, which govern the stability behavior of cracks, are computed and shown in curves for the cases of thermal conditions applied symmetrically and skew-symmetrically to the faces of an elliptical crack.

Fundamental Equations of Thermoelasticity

When the influences of both coupling and inertia are disregarded, the general thermal-stress problem separates into two distinct problems to be solved consecutively. The first is a problem in the theory of steady-state heat conduction which requires the temperatures $T(x,y,z)$ at every point of the body to satisfy the Laplace equation in three-dimensions:

$$\nabla^2 T(x,y,z) = 0 \quad . \tag{1}$$

Once the temperature distribution has been found, the resulting displacements and stresses may be obtained, respectively, from the Navier's displacement equation of static equilibrium

$$\frac{1}{1-2\nu} \nabla\nabla\cdot\underset{\sim}{u} + \nabla^2\underset{\sim}{u} = 2(\frac{1+\nu}{1-2\nu})\alpha\nabla T. \tag{2}$$

and the Duhamel-Neumann stress-displacement relation

$$\underset{\sim}{\sigma} = \mu\left(\nabla\underset{\sim}{u} + \underset{\sim}{u}\nabla + \frac{2}{1-2\nu}[\nabla\cdot\underset{\sim}{u} - (1+\nu)\alpha T]\underset{\sim}{I}\right). \tag{3}$$

in which $\underset{\sim}{u}$ is the displacement vector, $\underset{\sim}{\sigma}$ is the stress tensor and $\underset{\sim}{I}$ is the isotropic tensor. The gradient and Laplacian operators are denoted by ∇ and ∇^2, while μ, ν, and α designate the shear modulus, Poisson's ratio, and the coefficient of linear expansion of the solid whose mechanical and thermal properties are assumed to be isotropic and homogeneous.

If x,y,z stand for the Cartesian coordinates, the solution of eq. (2) for three-dimensional problems with geometric discontinuities on the plane z = 0 takes the form

$$\underset{\sim}{u} = \underset{\sim}{\phi} + z\nabla\psi^* \ , \ \psi^* = \psi^{**} + \psi. \tag{4}$$

where

$$\nabla^2\underset{\sim}{\phi} = 0, \ \nabla^2\psi^* = 0, \ \frac{\partial\psi}{\partial z} = \frac{1}{2}(\frac{1+\nu}{1-\nu})\alpha T. \tag{5}$$

and the vector displacement potential $\underset{\sim}{\phi}$ has components ϕ_x, ϕ_y, and ϕ_z. In eq. (4), ψ^{**} is the scalar displacement potential and ψ the thermoelastic potential. Hence, the dilatation becomes

$$\nabla\cdot\underset{\sim}{u} = \nabla\cdot\underset{\sim}{\phi} + \frac{\partial\psi^*}{\partial z} \ . \tag{6}$$

and the condition of equilibrium, eq. (2), requires that

$$\nabla \cdot \underset{\sim}{\phi} + \frac{\partial}{\partial z} [(3-4\nu)\psi^{**} - \psi] = 0. \tag{7}$$

For $T(x,y,z) = 0$, the above expressions reduce to those given in [6].

Using eqs. (3), (4) and (6), the components of the stress tensor $\underset{\sim}{\sigma}$, in terms of $\underset{\sim}{\phi}$ and ψ^*, may be obtained:

$$\frac{\sigma_{zz}}{2\mu} = \frac{\partial}{\partial z} (\phi_z + \psi^*) + \frac{\nu}{1-2\nu} (\nabla \cdot \underset{\sim}{\phi} + \frac{\partial \psi^*}{\partial z}) - 2(\frac{1-\nu}{1-2\nu}) \frac{\partial \psi^*}{\partial z} + z \frac{\partial^2 \psi^*}{\partial z^2} \tag{8a}$$

$$\frac{\tau_{xz}}{\mu} = \frac{\partial}{\partial x} (\phi_z + \psi^*) + \frac{\partial \phi_x}{\partial z} + 2z \frac{\partial^2 \psi^*}{\partial x \partial z} . \tag{8b}$$

$$\frac{\tau_{yz}}{\mu} = \frac{\partial}{\partial y} (\phi_z + \psi^*) + \frac{\partial \phi_y}{\partial z} + 2z \frac{\partial^2 \psi^*}{\partial y \partial z} . \tag{8c}$$

For the purpose of discussing symmetry conditions in the section to follow, the other stress components σ_{xx}, σ_{yy}, τ_{xy} are not of immediate interest.

Thermoelastic Problems with Planes of Discontinuities

In order to avoid unnecessary complication in the analysis, the surfaces of the plane of discontinuity or crack at $z = 0\pm$ are taken to be free from applied normal and shear stresses, since the method of solution when there are such applied stresses has already been treated in [6,8]. Moreover, the complete stress solution can be split up into two parts, which may be considered independently. The first part deals with the application of surface temperatures that are the same on both sides of the crack and the second considers the case where the temperatures on the upper surface, $z = 0^+$, of the crack are equal and opposite to those on the lower surface, $z = 0^-$. The corresponding stress systems will be referred to as the symmetrical part and skew-symmetrical part, respectively.

In the symmetrical part of the problem, the stresses and displacements, induced by thermal changes, will depend upon the variable z as follows:

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$$u, v, \sigma_{xx}, \sigma_{yy}, \sigma_{zz} \qquad ; \text{ even in } z \tag{9a}$$

$$w, \tau_{xz}, \tau_{yz} \qquad ; \text{ odd in } z \tag{9b}$$

These functions are required to be continuous outside the crack region on the plane $z = 0$ and the odd ones shown in eq. (9b) must be zero on such a plane. Furthermore, if the crack surfaces are free from applied stress, then $\sigma_{zz} = 0$. In view of symmetry, the condition $\tau_{xz} = \tau_{yz} = 0$ must hold everywhere on the plane $z = 0$. Hence, eqs. (8b) and (8c) give

$$\frac{\partial}{\partial x} (\phi_z + \psi^*) + \frac{\partial \phi_x}{\partial z} = 0, \ z = 0. \tag{10a}$$

$$\frac{\partial}{\partial y} (\phi_z + \psi^*) + \frac{\partial \phi_y}{\partial z} = 0, \ z = 0. \tag{10b}$$

At this point, it is convenient to introduce a harmonic function $f(x,y,z)$ such that both eqs. (7) and (10) are satisfied, i.e., by letting

$$\phi_x = (1-2\nu) \frac{\partial f}{\partial x} + \int_z^\infty \frac{\partial \psi}{\partial x} \, dz \ . \tag{11a}$$

$$\phi_y = (1-2\nu) \frac{\partial f}{\partial y} + \int_z^\infty \frac{\partial \psi}{\partial y} \, dz \ . \tag{11b}$$

$$\phi_z = -2(1-\nu) \frac{\partial f}{\partial z} \ , \ \psi^{**} = \frac{\partial f}{\partial z} \ . \tag{11c}$$

where

$$\nabla^2 f(x,y,z) = 0 \ .$$

The limits of the integrals in eqs. (11a) and (11b) have been chosen to satisfy the condition of regularity of the displacement at infinity. It follows that the displacements and stresses in the solid may be expressed in terms of two real harmonic functions $f(x,y,z)$ and $\psi(x,y,z)$. They are

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$$u = (1-2\nu)\,\frac{\partial f}{\partial x} + \int_z^\infty \frac{\partial \psi}{\partial x}\,dz + z\,\frac{\partial F}{\partial x} \quad . \tag{12a}$$

$$v = (1-2\nu)\,\frac{\partial f}{\partial y} + \int_z^\infty \frac{\partial \psi}{\partial y}\,dz + z\,\frac{\partial F}{\partial y} \quad . \tag{12b}$$

$$w = -2(1-\nu)\,\frac{\partial f}{\partial z} + z\,\frac{\partial F}{\partial z} \quad . \tag{12c}$$

and

$$\frac{\sigma_{xx}}{2\mu} = \frac{\partial^2 f}{\partial x^2} + 2\nu\,\frac{\partial^2 f}{\partial y^2} + \int_z^\infty \frac{\partial^2 \psi}{\partial x^2}\,dz - 2\,\frac{\partial \psi}{\partial z} + z\,\frac{\partial^2 F}{\partial x^2} \quad . \tag{13a}$$

$$\frac{\sigma_{yy}}{2\mu} = \frac{\partial^2 f}{\partial y^2} + 2\nu\,\frac{\partial^2 f}{\partial x^2} + \int_z^\infty \frac{\partial^2 \psi}{\partial y^2}\,dz - 2\,\frac{\partial \psi}{\partial z} + z\,\frac{\partial^2 F}{\partial y^2} \quad . \tag{13b}$$

$$\frac{\sigma_{zz}}{2\mu} = -\frac{\partial F}{\partial z} + z\,\frac{\partial^2 F}{\partial z^2} \quad . \tag{13c}$$

$$\frac{\tau_{xy}}{2\mu} = (1-2\nu)\,\frac{\partial^2 f}{\partial x \partial y} + \int_z^\infty \frac{\partial^2 \psi}{\partial x \partial y}\,dz + z\,\frac{\partial^2 F}{\partial x \partial y} \quad . \tag{13d}$$

$$\frac{\tau_{xz}}{2\mu} = z\,\frac{\partial^2 F}{\partial x \partial z} \quad . \tag{13e}$$

$$\frac{\tau_{yz}}{2\mu} = z\,\frac{\partial^2 F}{\partial y \partial z} \quad . \tag{13f}$$

in which

$$F = \frac{\partial f}{\partial z} + \psi \quad .$$

Similarly, the skew-symmetrical part of the problem may be formulated by having

$$u,\; v,\; \sigma_{xx},\; \sigma_{yy},\; \sigma_{zz} \qquad ;\; \text{odd in } z \tag{14a}$$

$$w,\; \tau_{xz},\; \tau_{yz} \qquad ;\; \text{even in } z \tag{14b}$$

to be continuous across the plane z = 0 with the exception of the crack

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region. The odd quantities in eq. (14a) must again vanish on the plane of continuity. On the crack surfaces, $\tau_{xz} = \tau_{yz} = 0$. Note that in the case of skew-symmetry $\sigma_{zz} = 0$ on the entire plane of $z = 0$, i.e.,

$$\frac{\partial}{\partial z} (\phi_z + \psi^{**}) + \frac{\nu}{1-\nu}\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y}\right) = \frac{\partial \psi}{\partial z} \quad . \tag{15}$$

Eliminating the function ψ by means of eqs. (7) and (15) yields

$$\frac{1}{1-\nu}\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial \phi_y}{\partial y}\right) + 2 \frac{\partial \psi^{**}}{\partial z} = 0 \quad . \tag{16}$$

Now, let $g(x,y,z)$ and $h(x,y,z)$ be two harmonic functions related to ϕ_x and ϕ_y by

$$\phi_x = -2(1-\nu) \frac{\partial g}{\partial z} \quad , \quad \phi_y = -2(1-\nu) \frac{\partial h}{\partial z} \quad . \tag{17}$$

where

$$\nabla^2 g(x,y,z) = 0 \quad , \quad \nabla^2 h(x,y,z) = 0 \quad .$$

It follows that

$$\psi^{**} = \frac{\partial g}{\partial x} + \frac{\partial h}{\partial y} \quad . \tag{18}$$

and

$$\phi_z = \psi - (1-2\nu) \psi^{**} \quad . \tag{19}$$

In a straightforward manner, the displacements are found as

$$u = -2(1-\nu) \frac{\partial g}{\partial z} + z \frac{\partial G}{\partial x} \quad . \tag{20a}$$

$$v = -2(1-\nu) \frac{\partial h}{\partial z} + z \frac{\partial G}{\partial y} \quad . \tag{20b}$$

$$w = -2(1-\nu)\left(\frac{\partial g}{\partial x} + \frac{\partial h}{\partial y}\right) + G + z \frac{\partial G}{\partial z} \quad . \tag{20c}$$

and the stresses are

$$\frac{\sigma_{xx}}{2\mu} = 2(1-\nu)\frac{\partial^2 h}{\partial y \partial z} - 2\frac{\partial G}{\partial z} + z\frac{\partial^2 G}{\partial x^2} \quad . \tag{21a}$$

$$\frac{\sigma_{yy}}{2\mu} = 2(1-\nu)\frac{\partial^2 g}{\partial x \partial z} - 2\frac{\partial G}{\partial z} + z\frac{\partial^2 G}{\partial y^2} \quad . \tag{21b}$$

$$\frac{\sigma_{zz}}{2\mu} = z\frac{\partial^2 G}{\partial z^2} \quad . \tag{21c}$$

$$\frac{\tau_{xy}}{2\mu} = -(1-\nu)\frac{\partial}{\partial z}\left(\frac{\partial g}{\partial y} + \frac{\partial h}{\partial x}\right) + z\frac{\partial^2 G}{\partial x \partial y} \quad . \tag{21d}$$

$$\frac{\tau_{xz}}{2\mu} = (1-\nu)\frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y} - \frac{\partial h}{\partial x}\right) + \frac{\partial G}{\partial x} + z\frac{\partial^2 G}{\partial x \partial z} \quad . \tag{21e}$$

$$\frac{\tau_{yz}}{2\mu} = (1-\nu)\frac{\partial}{\partial x}\left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}\right) + \frac{\partial G}{\partial y} + z\frac{\partial^2 G}{\partial y \partial z} \quad . \tag{21f}$$

where

$$G = \frac{\partial g}{\partial x} + \frac{\partial h}{\partial y} + \psi \quad . \tag{22}$$

The above formulation places no restriction on the geometry of the planes of discontinuities. However, for the sake of definiteness, the subsequent work will be concerned with a plane crack in the shape of an ellipse.

Temperature Distribution in an Infinite Solid

Consider the problem of an infinite solid the interior of which is exposed to uniform temperatures over a region occupied by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad , \quad z = 0 \quad .$$

Hence, it is expedient to solve this problem in ellipsoidal coordinates (ξ, η, ζ), which are related to the rectangular coordinates (x, y, z) of any

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point by [9]

$$a^2(a^2-b^2)x^2 = (a^2+\xi)(a^2+\eta)(a^2+\zeta) \ . \tag{23a}$$

$$b^2(b^2-a^2)y^2 = (b^2+\xi)(b^2+\eta)(b^2+\zeta) \ . \tag{23b}$$

$$a^2b^2z^2 = \xi\eta\zeta \ . \tag{23c}$$

where

$$\infty > \xi \geqslant 0 \geqslant \eta \geqslant -b^2 \geqslant \zeta \geqslant -a^2 \ .$$

In the plane $z = 0$, the inside of the ellipse is given by $\xi = 0$, and the out-
side by $\eta = 0$.

Let the temperature distribution $T(x,y,z)$ in the solid to be an even
function of z vanishing at infinity. The boundary conditions of interest
are

$$T = -T_0 \ , \quad \xi = 0 \ . \tag{24a}$$

$$\frac{\partial T}{\partial z} = 0 \ , \quad \eta = 0 \ . \tag{24b}$$

where T_0 is a constant. The solution of this problem is well known since
the temperature $T(x,y,z)$ is equivalent to the velocity potential of a per-
fect fluid passing through an elliptic aperture of a rigid partition[5].
Thus, it can easily be shown that

$$T(x,y,z) = - \frac{aT_0}{2K(k)} \int_\xi^\infty \frac{ds}{\sqrt{Q(s)}} = -T_0 \frac{u}{K(k)} \ . \tag{25}$$

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[5]See, for example, [9], p. 150, eq. (1).

Here, $K(k)$ is the complete elliptical integral of the first kind with argument $k^2 = 1 - (b/a)^2$ and

$$Q(s) = s(a^2+s)(b^2+s) \quad .$$

The variable u is associated with the Jacobian elliptic functions snu, cnu, ---, and should not be confused with the x-component of the displacement vector $\underset{\sim}{u}$. The relationship between u and the ellipsoidal coordinate ξ is

$$\xi = a^2 \left(\frac{cn\ u}{sn\ u}\right)^2 = a^2 (sn^{-2}u-1) \quad .$$

From eq. (25), the temperature gradient may be computed:

$$\frac{\partial T}{\partial z} = \frac{T_0}{bK(k)} \cdot \frac{\left[\eta\zeta(a^2+\xi)(b^2+\xi)\right]^{1/2}}{(\xi-\eta)(\xi-\zeta)} \quad . \tag{26}$$

On the plane $z = 0$, eqs. (25) and (26) provide the correct boundary conditions as

$$T = \begin{cases} -T_0 \ , \ \xi = 0 \\ \\ -T_0 \frac{u}{K(k)} \ , \ \eta = 0 \end{cases} \quad ; \quad \frac{\partial T}{\partial z} = \begin{cases} \frac{T_0}{bK(k)} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-1/2} \ , \ \xi = 0 \\ \\ 0 \ , \ \eta = 0 \ . \end{cases}$$

Substituting eq. (25) into the third expression in eq. (5) gives

$$\frac{\partial \psi}{\partial z} = -\left(\frac{1+\nu}{1-\nu}\right) \frac{a\alpha T_0}{4K(k)} \int_\xi^\infty \frac{ds}{\sqrt{Q(s)}} \quad . \tag{27}$$

The explicit expression of $\psi(x,y,z)$ is not required for this part of the problem. Terms containing the derivatives of $\psi(x,y,z)$, such as those in eqs. (12) and (13), will be found subsequently.

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If the elliptical region of exposure is maintained at a uniform temperature gradient Q_0, then the temperature $T(x,y,z)$ is an odd function of z. The boundary conditions are

$$\frac{\partial T}{\partial z} = -Q_0 \ , \ \xi = 0 \ . \tag{28a}$$

$$T = 0 \qquad , \ \eta = 0 \ . \tag{28b}$$

Aside from a multiplying constant, $T(x,y,z)$ is identical with the velocity potential for axial flow past an elliptic disk in an infinite fluid[6]. From this hydrodynamical analogy, it is found that

$$T(x,y,z) = \frac{ab^2 Q_0}{2E(k)} \cdot z \int_\xi^\infty \frac{ds}{s\sqrt{Q(s)}} = \frac{Q_0}{E(k)} \left[\frac{snu \ dnu}{cn \ u} - E(u) \right] \cdot z \ . \tag{29}$$

where

$$E(u) = \int_0^u dn^2 t \ dt \ .$$

and $E(k)$ is the complete elliptical integral of the second kind. Using eqs. (28) and (5), the thermoelastic potential can be determined from the conditions

$$\frac{\partial^2 \psi}{\partial z^2} = -\left(\frac{1+\nu}{1-\nu} \right) \frac{\alpha Q_0}{2} \ , \ \xi = 0 \ .$$

$$\frac{\partial \psi}{\partial z} = 0 \ , \ \eta = 0 \ .$$

which can be written in the equivalent form

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[6]See [9], p. 144, eq. (8).

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$$\psi = H(x,y) \ , \ \xi = 0 \ .$$

$$\frac{\partial \psi}{\partial z} = 0, \ \eta = 0 \ .$$

The function $H(x,y)$ may be considered to be any particular solution of the equation

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = (\frac{1+\nu}{1-\nu}) \frac{\alpha Q_0}{2} \ .$$

Furthermore, if $H(x,y)$ is sufficiently smooth, a suitable solution for the thermoelastic potential may be taken in the form of the Newtonian potential of a simple layer with a continuously differentiable density. Without going into details, it can be verified that putting

$$\psi(x,y,z) = (\frac{1+\nu}{1-\nu}) \frac{ab^2 \alpha Q_0}{8E(k)} \int_{\xi}^{\infty} \left(\frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{s} - 1 \right) \frac{ds}{\sqrt{Q(s)}} \ . \tag{30}$$

into $\alpha T = 2(\frac{1-\nu}{1+\nu}) \frac{\partial \psi}{\partial z}$ results in eq. (29).

Having obtained the thermoelastic potentials given by eqs. (27) and (30), a more detailed treatment of the thermoelastic problem is in order.

The Elliptical Crack Problem

Let the surfaces of a flat elliptical crack in an infinite solid be opened by the application of uniform temperature T_0 and temperature gradient Q_0 as described in the previous section. In the absence of mechanical and thermal disturbances at sufficiently large distances away from the crack, the displacements and stresses are assumed to vanish as z approaches infinity. From the knowledge of $\psi(x,y,z)$ or its derivative with respect to z, the complete solution of the present problem requires the evaluation of one function $f(x,y,z)$ for the symmetric part and two functions $g(x,y,z)$ and $h(x,y,z)$ for the skew-symmetric part.

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Suppose that the prescribed temperature are constant across the upper surface of the elliptical crack and are exactly the same as those across the lower surface given by eqs. (24). In addition, the corresponding mechanical conditions are

$$\sigma_{zz} = 0, \ \xi = 0 \ ; \ w = 0, \ \eta = 0 \ .$$

Thus, eqs. (12c) and (13c) give

$$\frac{\partial F}{\partial z} = 0, \ \xi = 0 \ . \tag{31a}$$

$$\frac{\partial f}{\partial z} = 0, \ \eta = 0 \ . \tag{31b}$$

An appropriate form of the function $f(x,y,z)$ is [6]

$$f(x,y,z) = 1/2 \ A \int_{\xi}^{\infty} \left(\frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{s} - 1 \right) \frac{ds}{\sqrt{Q(s)}} \ . \tag{32}$$

which satisfies eq. (31a) and the constant A is obtained from eq. (31b) given by

$$A = \left(\frac{1+\nu}{1-\nu} \right) \cdot \frac{ab^2 \alpha T_0}{4E(k)} \ . \tag{33}$$

To find the displacements u and v, it is necessary to evaluate

$$\frac{\partial^2 \psi}{\partial x \partial z} = \left(\frac{1+\nu}{1-\nu} \right) \cdot \frac{a \alpha T_0}{8K(k)} \cdot \frac{x}{h_1^2(a^2+\xi)\sqrt{Q(\xi)}} \ .$$

obtained from eq. (27). Here,

$$h_1^2 = \frac{(\xi-\eta)(\xi-\zeta)}{4Q(\xi)} \ .$$

Upon integration with respect to the variable z yields

$$\frac{\partial \psi}{\partial x} = -\left(\frac{1+\nu}{1-\nu} \right) \cdot \frac{a \alpha T_0}{8K(k)} \int_{z}^{\infty} \frac{x \ dz}{h_1^2(a^2+\xi)\sqrt{Q(\xi)}} \ . \tag{34}$$

where the variable of integration may be changed from z to ξ by the relation [9]

$$\frac{dz}{h_1^2} = \frac{2\xi\ d\xi}{z} \quad .$$

Introducing s as a dummy variable of integration, eq. (34) becomes

$$\frac{\partial\psi}{\partial x} = -(\frac{1+\nu}{1-\nu}) \cdot \frac{a\alpha T_0}{4K(k)} \cdot x \int_\xi^\infty \frac{s\ ds}{(a^2+s)\ z(s)\sqrt{Q(s)}} \quad .$$

where z(s) is determined from

$$\frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{s} -1 = 0 \quad . \tag{35}$$

Integrating eq. (34) once more gives

$$\int_z^\infty \frac{\partial\psi}{\partial x}\ dz = -(\frac{1+\nu}{1-\nu}) \cdot \frac{a\alpha T_0}{4K(k)} \cdot x \int_z^\infty [\ \int_\xi^\infty J(s;x,y)\ ds\ ]\ dz \quad . \tag{36}$$

in which

$$J(s;x,y) = \frac{(1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s})^{-1/2}}{(a^2+s)^{3/2}\ (b^2+s)^{1/2}} \quad .$$

Since integrals of the form

$$\int_z^\infty dz \int_\xi^\infty \lambda(s)\ ds \quad . \tag{37}$$

may be evaluated from the formula

$$-\int_z^\infty dz \int_\xi^\infty \lambda(s)\ ds = \int_\xi^\infty [z - z(s)]\ \lambda(s)\ ds \quad . \tag{38}$$

eq. (36) takes the form[7]

$$\int_z^\infty \frac{\partial \psi}{\partial x} \, dz = \left(\frac{1+\nu}{1-\nu}\right) \cdot \frac{a\alpha T_0}{4K(k)} \cdot x \, [z \, I_1 - I_2] \, . \tag{39a}$$

where

$$I_1 = \int_\xi^\infty (a^2+s)^{-3/2} (b^2+s)^{-1/2} \left[1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s} \right]^{-1/2} ds \, .$$

$$I_2 = \int_\xi^\infty s^{1/2} (a^2+s)^{-3/2} (b^2+s)^{-1/2} \, ds \, .$$

In a similar manner, it can be shown that

$$\int_z^\infty \frac{\partial \psi}{\partial y} \, dz = \left(\frac{1+\nu}{1-\nu}\right) \cdot \frac{a\alpha T_0}{4K(k)} \, y \, [z I_3 - I_4] \, . \tag{39b}$$

where

$$I_3 = \int_\xi^\infty (a^2+s)^{-1/2} (b^2+s)^{-3/2} \left[1 - \frac{x^2}{a^2+s} \quad \frac{y^2}{b^2+s} \right]^{-1/2} ds \, .$$

$$I_4 = \int_\xi^\infty s^{1/2} (a^2+s)^{-1/2} (b^2+s)^{-3/2} \, ds \, .$$

The displacements in the plane $z = 0$ are found as[8]

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[7]The evalu.·ion of eq. (39a) is intimately related to the elliptical punch problem which was solved only in part by Green and Sneddon [8]. Their solution can be completed by following the steps going from eq. (27) to (39). Incidentally, a complete solution to the elliptical punch problem has been given by Galin [11].

[8]See Appendix (1) for the results of integrals.

$$u = -\frac{A}{b^2} \cdot x \left[(1-2\nu)b^2 \int_{\xi}^{\infty} \frac{ds}{(a^2+s)\sqrt{Q(s)}} + \frac{E(k)}{K(k)} \int_{\xi}^{\infty} \frac{sds}{(a^2+s)\sqrt{Q(s)}} \right]$$

$$= -\frac{2A}{abk^2} \cdot x \left\{ (1-2\nu) \left(\frac{b}{a}\right)^2 [u-E(u)] + \frac{E(k)}{K(k)} [E(u) - k'^2 u] \right\} . \tag{40a}$$

$$v = -\frac{A}{b^2} \cdot y \left[(1-2\nu)b^2 \int_{\xi}^{\infty} \frac{ds}{(b^2+s)\sqrt{Q(s)}} + \frac{E(k)}{K(k)} \int_{\xi}^{\infty} \frac{sds}{(b^2+s)\sqrt{Q(s)}} \right]$$

$$= -\frac{2A}{abk^2} \cdot y \left\{ (1-2\nu) \left[E(u) - \left(\frac{b}{a}\right)^2 u - k^2 \cdot \frac{snu\ cnu}{dn\ u}\right] \right.$$

$$\left. + \frac{E(k)}{K(k)} \left[u-E(u) + k^2 \cdot \frac{snu\ cnu}{dn\ u}\right] \right\} . \tag{40b}$$

$$w = 2(1-\nu) A \lim_{z\to 0} \left[z \cdot \int_{\xi}^{\infty} \frac{ds}{S\sqrt{Q(s)}} \right]$$

$$= \frac{4(1-\nu)A}{ab^2} \lim_{z\to 0} \left\{ z \left[\frac{snu\ dnu}{cn\ u} - E(u)\right] \right\} . \tag{40c}$$

Now, let u_0, v_0, and w_0 be the displacements of the crack surfaces. Then as $\xi \to 0$:

$$sn\ u \to 1, \quad cn\ u \to 0, \quad dn\ u \to \frac{b}{a}, \quad E(u) \to E(k) .$$

and eqs. (40) simply reduce to

$$u_0 = -\frac{2Ab}{a^3k^2} E(k) \cdot x \left\{ (1-2\nu) \left[\frac{K(k)}{E(k)} -1\right] + \left(\frac{a}{b}\right)^2 \frac{E(k)}{K(k)} - 1 \right\} . \tag{41a}$$

$$v_0 = -\frac{2Ab}{a^3k^2} E(k) \cdot y \left\{ (1-2\nu) \left[\left(\frac{a}{b}\right)^2 - \frac{K(k)}{E(k)}\right] + 1 - \frac{E(k)}{K(k)} \right\} . \tag{41b}$$

$$w_0 = \frac{4(1-\nu)A}{ab} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2} . \tag{41c}$$

In the special case of a penny-shaped crack, $a = b$, $E(k) = K(k) = \pi/2$, eqs. (41) may be further simplified as

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = -\frac{(1+\nu)\alpha T_0}{2} \begin{bmatrix} x \\ y \end{bmatrix} , \quad w_0 = \frac{2(1+\nu)a\alpha T_0}{\pi} \sqrt{1-\rho^2} . \tag{42}$$

where $\rho = r/a$ and $r^2 = x^2 + y^2$. The normal component of the displacement w_0 of the crack surface agree with eq. (90) in [3] only when the integrals in [3] are evaluated properly[9]. As a consequence, the variation of w_0 with ρ shown by Fig. 3 in [3] should also be changed accordingly.

The problem of a uniform steady heat flow disturbed by an insulated elliptical crack is equivalent to the one of assigning uniform temperature gradients to the crack surfaces. The thermal conditions are shown in eqs. (28) and

$$\tau_{xz} = \tau_{yz} = 0, \, \xi = 0 : u = v = 0, \, \eta = 0 .$$

must be satisfied on the plane $z = 0$. From eqs. (20a), (20b), (21e), and (21f), these conditions may be put into the forms

$$\frac{\partial^2 g}{\partial z^2} + \nu \frac{\partial}{\partial y} \left(\frac{\partial g}{\partial y} - \frac{\partial h}{\partial x} \right) = \frac{\partial \psi}{\partial x} , \, \xi = 0 . \tag{43a}$$

$$\frac{\partial^2 h}{\partial z^2} + \nu \frac{\partial}{\partial x} \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y} \right) = \frac{\partial \psi}{\partial y} , \, \xi = 0 . \tag{43b}$$

---

[9]The corrected expressions in Olesiak and Sneddon's paper [3], p. 253, should read as

$$\psi(\eta) = \frac{2(1+\nu)\alpha\theta_0 a}{\pi} \left(\frac{\sin \eta - \eta \cos \eta}{\eta^2} \right) . \tag{88}$$

with

$$\int_0^t s \, Q(s) \, ds = \frac{2\theta_0}{\pi a} \left[1 - \sqrt{1-t^2} \right] , \quad C = \frac{2\theta_0}{\pi a} .$$

and

$$W(\rho) = \frac{2\theta_0}{\pi a} \left[(1-\rho^2)^{1/2} - \log (1 + \sqrt{1-\rho^2}) \right] .$$

Hence,

$$W_0 = L_0 (1-\rho^2)^{1/2} . \tag{90}$$

where

$$L_0 = \frac{2(1+\nu)\alpha\theta_0 a}{\pi} . \tag{91}$$

and θ_0 corresponds to T_0 in the present work.

---

where the right hand side of eqs. (43) are known quantities from eq. (30), and

$$\frac{\partial g}{\partial z} = 0 , \frac{\partial h}{\partial z} = 0 , \eta = 0 . \tag{44}$$

Both $g(x,y,z)$ and $h(x,y,z)$ are even functions of z. The character of eqs. (43) suggests that the problem can be readily solved by putting[10]

$$\frac{\partial g}{\partial z} = \left(x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} \right) \Lambda , \tag{45a}$$

$$\frac{\partial h}{\partial z} = \left(y \frac{\partial}{\partial z} - \frac{\partial}{\partial y} \right) \Omega . \tag{45b}$$

Equations (45a) and (45b) represent the velocity potential of a rigid elliptical disk rotating about the y-axis and x-axis, respectively[11].

---

[10]If $H(x,y,z)$ is a harmonic function, i.e., $\nabla^2 H(x,y,z) = 0$, then $(x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x}) H$ and $(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}) H$ are also harmonic.
[11]See [10], p. 145.

By virtue of eqs. (45), eqs. (20) render

$$u = -2(1-\nu) \, x \, \frac{\partial \Lambda}{\partial z} + z \, \frac{\partial}{\partial x} \left[\, \Gamma + 2(1-\nu)\Lambda \, \right] \quad , \tag{46a}$$

$$v = -2(1-\nu) \, y \, \frac{\partial \Omega}{\partial z} + z \, \frac{\partial}{\partial y} \left[\, \Gamma + 2(1-\nu)\Omega \, \right] \quad , \tag{46b}$$

$$w = 2(1-\nu) \, \psi - (1-2\nu) \, \Gamma + z \, \frac{\partial \Gamma}{\partial z} \quad . \tag{46c}$$

Equations (21) may also be expressed in terms of the functions $\Lambda(x,y,z)$ and $\Omega(x,y,z)$:

$$\frac{\sigma_{xx}}{2\mu} = 2(1-\nu) \, \frac{\partial}{\partial z} \left(y \, \frac{\partial}{\partial y} + 1 \right) \Omega - 2 \, \frac{\partial \Gamma}{\partial z} + z \left[\frac{\partial^2 \Gamma}{\partial x^2} - 2(1-\nu) \, \frac{\partial^2 \Omega}{\partial y^2} \right] \quad . \tag{47a}$$

$$\frac{\sigma_{yy}}{2\mu} = 2(1-\nu) \, \frac{\partial}{\partial z} \left(x \, \frac{\partial}{\partial x} + 1 \right) \Lambda - 2 \, \frac{\partial \Gamma}{\partial z} + z \left[\frac{\partial^2 \Gamma}{\partial y^2} - 2(1-\nu) \, \frac{\partial^2 \Lambda}{\partial x^2} \right] \quad . \tag{47b}$$

$$\frac{\sigma_{zz}}{2\mu} = z \, \frac{\partial^2 \Gamma}{\partial z^2} \quad . \tag{47c}$$

$$\frac{\tau_{xy}}{2\mu} = -(1-\nu) \, \frac{\partial}{\partial z} \left(x \, \frac{\partial \Lambda}{\partial y} + y \, \frac{\partial \Omega}{\partial x} \right) + z \, \frac{\partial^2}{\partial x \partial y} \left[\Gamma + (1-\nu)(\Lambda+\Omega) \right] \quad . \tag{47d}$$

$$\frac{\tau_{xz}}{2\mu} = (1-\nu) \, \frac{\partial}{\partial y} \left(x \, \frac{\partial \Lambda}{\partial y} - y \, \frac{\partial \Omega}{\partial x} \right) + \frac{\partial}{\partial x} \left(z \, \frac{\partial}{\partial z} + 1 \right) \Gamma$$
$$+ (1-\nu) \, \frac{\partial^3}{\partial x \partial y^2} \left[\int_z^\infty z(\Lambda-\Omega) \, dz \right] \quad . \tag{47e}$$

$$\frac{\tau_{yz}}{2\mu} = (1-\nu) \, \frac{\partial}{\partial x} \left(y \, \frac{\partial \Omega}{\partial x} - x \, \frac{\partial \Lambda}{\partial y} \right) + \frac{\partial}{\partial y} \left(z \, \frac{\partial}{\partial z} + 1 \right) \Gamma$$
$$- (1-\nu) \, \frac{\partial^3}{\partial x^2 \partial y} \left[\int_z^\infty z(\Lambda-\Omega) \, dz \right] \quad . \tag{47f}$$

where the following contractions has been made:

$$\Gamma = \psi + \Lambda + x \, \frac{\partial \Lambda}{\partial x} + y \, \frac{\partial \Omega}{\partial y} + z \, \frac{\partial \Omega}{\partial z} + \frac{\partial^2}{\partial x^2} \left[\int_z^\infty z(\Lambda-\Omega) \, dz \right] \quad .$$

From eqs. (30), (43), and (45) follow then

$$\frac{\partial}{\partial z} (x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x}) \Lambda + \nu \left(\frac{\partial}{\partial y} (x \frac{\partial \Lambda}{\partial y} - y \frac{\partial \Omega}{\partial x}) + \frac{\partial^3}{\partial x \partial y^2} \left[\int_z^\infty z(\Lambda-\Omega) \, dz \right] \right)$$

$$= (\frac{1+\nu}{1-\nu}) \frac{\alpha Q_0}{2k^2} \left[\frac{K(k)}{E(k)} - 1 \right] (\frac{b}{a})^2 \cdot x \ , \ \xi = 0 \quad . \tag{48a}$$

$$\frac{\partial}{\partial z} (y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}) \Omega + \nu \left(\frac{\partial}{\partial x} (y \frac{\partial \Omega}{\partial x} - x \frac{\partial \Lambda}{\partial y}) - \frac{\partial^3}{\partial x^2 \partial y} \left[\int_z^\infty z(\Lambda-\Omega) \, dz \right] \right)$$

$$= (\frac{1+\nu}{1-\nu}) \frac{\alpha Q_0}{2k^2} \left[1 - (\frac{b}{a})^2 \frac{K(k)}{E(k)} \right] \cdot y \ , \ \xi = 0 \quad . \tag{48b}$$

Taking $\Lambda(x,y,z)$ and $\Omega(x,y,z)$ in the forms

$$\begin{bmatrix} \Lambda \\ \\ \Omega \end{bmatrix} = 1/2 \begin{bmatrix} B \\ \\ C \end{bmatrix} \int_\xi^\infty (\frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{s} - 1) \frac{ds}{\sqrt{Q(s)}} \quad . \tag{49}$$

the constants B and C may be solved from eqs. (48), which yield

$$B = \frac{m_1 q_2 - m_2 q_1}{p_1 q_2 - p_2 q_1} \ , \quad C = \frac{m_1 p_2 - m_2 p_1}{q_1 p_2 - q_2 p_1} \quad . \tag{50}$$

The constants p_j, q_j and m_j (j=1,2) are

$$p_1 = -\frac{E(k)}{b^2} + \frac{1}{a^2-b^2} \left(E(k) - K(k) + \nu \left[(\frac{a}{b})^2 E(k) - K(k) \right. \right.$$
$$\left. \left. - \frac{(a^2+b^2)K(k) - 2a^2 E(k)}{a^2 - b^2} \right] \right) \ ,$$

$$q_1 = \frac{\nu}{a^2-b^2} \left[E(k) - K(k) + \frac{(a^2+b^2)K(k) - 2a^2 E(k)}{a^2 - b^2} \right] \quad .$$

$$m_1 = (\frac{1+\nu}{1-\nu}) \cdot \frac{a\alpha Q_0}{4k^2} \left[\frac{K(k)}{E(k)} - 1 \right] (\frac{b}{a})^2 \quad .$$

and

$$p_2 = \frac{\nu}{a^2-b^2} \left[K(k) - (\frac{a}{b})^2 E(k) + \frac{(a^2+b^2)K(k) - 2a^2 E(k)}{a^2 - b^2} \right] \quad .$$

$$q_2 = \frac{E(k)}{b^2} + \frac{1}{a^2-b^2} \left\{ k(k) - \left(\frac{a}{b}\right)^2 E(k) + \nu \left[K(k) - E(k) \right. \right.$$

$$\left. \left. + \frac{(a^2+b^2)K(k) - 2a^2E(k)}{a^2 - b^2} \right] \right\},$$

$$m_2 = \left(\frac{1+\nu}{1-\nu}\right) \cdot \frac{a\alpha Q_0}{4k^2} \left[1 - \left(\frac{b}{a}\right)^2 \frac{K(k)}{E(k)} \right].$$

The solution is essentially complete and is a complicated function of the material constants and the geometry of the problem.

A straightforward calculation employing eqs. (46) gives the displacement components[12] for $z = 0$

$$\begin{Bmatrix} u \\ \\ v \end{Bmatrix} = -2(1-\nu) \begin{Bmatrix} Bx \\ \\ Cy \end{Bmatrix} \lim_{z\to 0} \cdot \frac{2}{ab^2} \left\{ z \left[\frac{snu\ dnu}{cn\ u} - E(u) \right] \right\} \tag{51a}$$

$$a^3k^2w = [\ u-E(u)\ ] \left\{ \left[\left(\frac{1+\nu}{1-\nu}\right) \frac{ab^2\alpha Q_0}{4E(k)} - (1-2\nu) \right. \right.$$

$$\left. \left. \left[3B + \left(1 - \frac{2}{k^2}\right) (B-C) \right] \right] x + \frac{y^2}{k^2} \right\}.$$

$$+ \left\{ \left(\frac{1+\nu}{1-\nu}\right) \frac{ab^2\alpha Q_0}{4E(k)} - (1-2\nu) \left[3B + 2C + \left(\frac{b}{ak}\right)^2 (B-C) \right] \right\} \left(\frac{a}{b}\right)^2 \left[E(u) - k^2 \cdot \right.$$

$$\left. \frac{snu\ cnu}{dn\ u} \right] - u \right\} \cdot y^2$$

$$- \left\{ a^2k^2 \left[\left(\frac{1+\nu}{1-\nu}\right) \frac{ab^2\alpha Q_0}{4E(k)} - (1-2\nu) B \right] + (1-2\nu) (B-C) x^2 \right\} u$$

$$+ (1-2\nu) (B-C) x^2 sn\ u\ \ cn\ u\ \ dn\ u\ . \tag{51b}$$

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[12] The results of the integrals are tabulated in Appendix (2).

In the same way as before by setting $\xi = 0$, the displacements of the crack surfaces are found:

$$\begin{Bmatrix} u_0 \\ \\ v_0 \end{Bmatrix} = -\frac{4(1-\nu)}{ab} \begin{Bmatrix} Bx \\ \\ Cy \end{Bmatrix} \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2}$$ (52a)

$$a^3 k^2 w_0 = [\, K(k) - E(k) \,] \left\{ \left[\left(\frac{1+\nu}{1-\nu}\right) \frac{ab^2 \alpha Q_0}{4E(k)} - (1-2\nu) \right. \right.$$

$$\left. [\, 3B + (1 - \frac{2}{k^2})(B-C) \,] \right] x^2 + \frac{y^2}{k^2} \Bigg\} \cdot$$

$$+ \left\{ \left(\frac{1+\nu}{1-\nu}\right) \frac{ab^2 \alpha Q_0}{4E(k)} - (1-2\nu) \left[3B + 2C + \left(\frac{b}{ak}\right)^2 (B-C) \right] \right\}$$

$$[\, (\frac{a}{b})^2 E(k) - K(k) \,] \cdot y^2$$

$$- \left\{ a^2 k^2 \left[\left(\frac{1+\nu}{1-\nu}\right) \frac{ab^2 \alpha Q_0}{4E(k)} - (1-2\nu) B \right] + (1-2\nu)(B-C) x^2 \right\} K(k) \quad .$$ (52b)

In the limit as $a = b$, eqs. (50) shows that

$$B = C = - \left(\frac{1+\nu}{1-\nu}\right) \cdot \frac{a^3 \alpha Q_0}{6\pi} \quad .$$

Because of rotational symmetry of the circular crack problem, the displacements u and v may be combined as $(u_0^2 + v_0^2)^{1/2}$ to give the radial component u_r:

$$3\sqrt{\frac{\pi}{2}} \cdot \frac{u_r}{(1+\nu)a^2 \alpha Q_0} = \sqrt{\frac{2}{\pi}} \; \rho \cdot \sqrt{1-\rho^2} \; , \; \rho < 1 \quad .$$ (53)

The normal displacement w_0 is[13]

$$\left(\frac{1-\nu}{1+\nu}\right) \cdot \frac{w_0}{a^2\alpha Q_0} = \frac{1}{4} \left[(1-\nu)\, \rho^2 - \frac{2}{3}(2-\nu) \right] , \quad \rho < 1 \quad . \tag{54a}$$

and

$$\left(\frac{1-\nu}{1+\nu}\right) \cdot \frac{w_0}{a^2\alpha Q_0} = \frac{1}{2\pi} \left(\left[(1-\nu)\, \rho^2 - \frac{2}{3}(2-\nu) \right] \sin^{-1}\left(\frac{1}{\rho}\right) \right.$$

$$\left. - (1-\nu)\, \sqrt{\rho^2-1} \right) \quad \rho > 1 \quad . \tag{54b}$$

Eqs. (53) and (54) check with eqs. (26) and (27) in [5], respectively, when the expression for w_0 in [5] is corrected. The temperature gradient Q_0 corresponds to τ in [5].

The computation of thermal stresses will be considered in the next section.

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[13]For $\rho > 1$, w_0 in Florence and Goodier's paper [5] should be corrected.

$$\sqrt{2\pi} \left(\frac{1-\nu}{1+\nu}\right) \frac{w_0}{a^2\alpha Q_0} = -\frac{1}{3} \sqrt{\frac{2}{\pi}} \frac{1}{\rho} \cdot {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{1}{\rho^2}\right)$$

$$+ \frac{2(1-2\nu)}{45\sqrt{2\pi}} \cdot \frac{1}{\rho^3} \cdot {}_2F_1\left(\frac{3}{2},\frac{3}{2};\frac{7}{2};\frac{1}{\rho^2}\right) , \quad \rho > 1 \quad .$$

where ${}_2F_1(a,b;c;z)$ is Gauss' hypergeometric series. Moreover, these hypergeometric series can be reduced to elementary functions by means of Gauss' recursion formulas [9] and some properties of the hypergeometric series. The results are

$$\frac{1}{\rho}\, {}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{5}{2};\frac{1}{\rho^2}\right) = \frac{3}{4}\left[\sqrt{\rho^2-1} - (\rho^2-2)\sin^{-1}\left(\frac{1}{\rho}\right) \right] ,$$

$$\frac{1}{\rho^3} \cdot {}_2F_1\left(\frac{3}{2},\frac{3}{2};\frac{7}{2};\frac{1}{\rho^2}\right) = \frac{45}{4}\left[\left(\rho^2 - \frac{2}{3}\right)\sin^{-1}\left(\frac{1}{\rho}\right) - \sqrt{\rho^2-1} \right] .$$

There is also a misprint in [5], eq. (27), where the factor $\pi/4$ should read as $1/4$.

Crack-Border Thermal Stresses and Stress-Intensity Factors

A knowledge of thermal stresses in a small region ahead of the crack is essential in the investigation of the stability behavior of cracks. For instance, the Griffith-Irwin theory of fracture [7] is based upon the concept that the onset of rapid crack extension occurs when the magnitude of crack-border stress field or simply the stress-intensity factor reaches some critical value. For a given configuration of the crack, this value will in general, depend upon the properties of the material such as the shear modulus, Poisson's ratio, and the coefficient of thermal expansion, etc. Hence, attention will be focused on the determination of stress-intensity factors $k_j(j=1,2,3)$, where each one of the k-factors refers to a particular mode of crack surface displacement. The elliptical crack problem will be used as an example.

The "opening mode" of crack extension, governed by k_1, arises in the symmetrical problem. For the mere purpose of finding k_1, it suffices to calculate σ_{zz} in the plane $z = 0$, outside the elliptical crack region. By way of eqs. (32) and (13c)[14]

$$
\frac{\sigma_{zz}}{2\mu} = A \left(\frac{2}{\xi^{1/2} (a^2+\xi)^{1/2} (b^2+\xi)^{1/2}} - \int_\xi^\infty \frac{(2s+a^2+b^2)ds}{(a^2+s)(b^2+s)\sqrt{Q(s)}} \right)
$$

$$
+ \left(\frac{1+\nu}{1-\nu}\right) \cdot \frac{a\alpha T_0}{4K(k)} \int_\xi^\infty \frac{ds}{\sqrt{Q(s)}} \quad .
$$

$$
= \left(\frac{1+\nu}{1-\nu}\right) \frac{\alpha T_0}{2} \left(\frac{1}{E(k)} \left[\frac{ab^2}{\xi^{1/2} (a^2+\xi)^{1/2} (b^2+\xi)^{1/2}} - E(u) + \frac{\text{snu cnu}}{\text{dn u}} \right] \right.
$$

$$
\left. + \frac{u}{K(k)} \right) \quad , \eta = 0 \quad . \tag{55}
$$

---

[14]Refer to Appendix (1) for solutions to integrals.

- 27 -

In the neighborhood of the crack border, the ellipsoidal coordinates ξ and ζ have the limits [15]

$$\xi = 2\ abr\ (a^2\ \sin^2\phi + b^2\ \cos^2\phi)^{-1/2} \quad , \tag{56a}$$

$$\zeta = -\ (a^2\ \sin^2\phi + b^2\ \cos^2\phi) \quad . \tag{56b}$$

which are valid for $\eta = 0$ and $r \ll 1$. The small distance r in the plane $z = 0$ is measured normal to the crack border and ϕ is the angle in the parametric equations of an ellipse. Substituting eq. (56a) into (55), the expansion of σ_{zz} for small values of r (or ξ) is

$$\sigma_{zz} = (\frac{1+\nu}{1-\nu})\ \frac{\mu\ \alpha T_0}{E(k)} \sqrt{\frac{b}{a}}\ (a^2\ \sin^2\phi + b^2\ \cos^2\phi)^{1/4}\ \frac{1}{\sqrt{2r}} + O(r^{1/2}) \quad . \tag{57}$$

The stress-intensity factor k_1 can now be extracted from eq. (57) as

$$k_1 = (\frac{1+\nu}{1-\nu})\ \frac{\mu\ \alpha T_0}{E(k)} \sqrt{\frac{b}{a}}\ (a^2\ \sin^2\phi + b^2\ \cos^2\phi)^{1/4} \quad , a > b \quad . \tag{58}$$

The variation of k_1 with the angle ϕ is plotted in Fig. 1 for $\nu = 1/3$ and different values of the ratio a/b. A glance at the curves shows that k_1 is always greatest at $\phi = 90°$, i.e., the intersection of the crack boundary with the minor axis of the ellipse. Thus, crack propagation, if it occurs, would first take place at the point $(0,b,0)$, and tend to produce a penny-shaped crack. This is a brief physical interpretation of the Griffith-Irwin concepts of fracture.

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[15]See [6], eqs. (52a) and (55).

The remaining strength of a solid whose continuity is interrupted by a circular crack may be estimated from eqs. (57) and (58) by setting $a = b$ and $\xi = r^2 - a^2$. Consequently[16]

$$\frac{\sigma_{zz}}{2\mu} = \left(\frac{1+\nu}{1-\nu}\right) \frac{\alpha T_0}{\pi} (\rho^2 - 1)^{-1/2} \qquad \rho > 1 \quad .$$

and   (59)

$$k_1 = \left(\frac{1+\nu}{1-\nu}\right) \frac{2\mu \alpha T_0}{\pi} \sqrt{a} \quad .$$

The stress-intensity factors k_2 and k_3 correspond, respectively, to the "edge-sliding" and "tearing" modes of fracture. Their evaluation calls for the expressions τ_{xz} and τ_{yz} as $z \to 0$. It is found from eqs. (47e), (47f) and (49) that[17]

$$
\begin{aligned}
\frac{\tau_{xz}}{2\mu} = {} & \left[\frac{(a^2+\xi)(a^2+\zeta)}{a^2 - b^2} \right]^{1/2} \Bigg(\left[\left(\frac{1+\nu}{1-\nu}\right) \frac{ab^2\alpha Q_0}{4E(k)} + B + \nu C \right] \int_\xi^\infty \frac{ds}{(a^2+s)\sqrt{Q(s)}} \\
& - B \left[\frac{2}{\sqrt{Q(\xi)}} - \int_\xi^\infty \frac{(2s+a^2+b^2)ds}{(a^2+s)(b^2+s)\sqrt{Q(s)}} + \nu \int_\xi^\infty \frac{ds}{(b^2+s)\sqrt{Q(s)}} \right] \\
& + \frac{2\nu(b^2+\zeta)}{\xi^{1/2}(\xi-\zeta)(b^2-a^2)} \left[B \left(\frac{a^2+\xi}{b^2+\xi}\right)^{1/2} \left(1 - \frac{\xi}{a^2+\xi}\right) - C \left(\frac{b^2+\xi}{a^2+\xi}\right)^{1/2} \left(1 - \frac{\xi}{b^2+\xi}\right) \right] \\
& + \nu(B - C) \int_\xi^\infty \frac{s\,ds}{(a^2+s)(b^2+s)\sqrt{Q(s)}} \Bigg) \quad , \quad \eta = 0 \quad . \qquad (60a)
\end{aligned}
$$

---

[16]The factor of $a/3$ should be taken out of eq. (92) in [3].

[17]The integrals are tabulated in Appendix (2).

$$\frac{\tau_{yz}}{2\mu} = \left[\frac{(b^2+\xi)(b^2+\zeta)}{b^2 - a^2} \right]^{1/2} \cdot \left(\left[(\frac{1+\nu}{1-\nu}) \frac{ab^2\alpha Q_0}{4E(k)} + C + \nu B \right] \int_\xi^\infty \frac{ds}{(b^2+s)\sqrt{Q(s)}} \right.$$

$$- C \left[\frac{2}{\sqrt{Q(\xi)}} - \int_\xi^\infty \frac{(2s+a^2+b^2)ds}{(a^2+s)(b^2+s)\sqrt{Q(s)}} + \nu \int_\xi^\infty \frac{ds}{(a^2+s)\sqrt{Q(s)}} \right]$$

$$- \frac{2\nu(a^2+\zeta)}{\xi^{1/2}(\xi-\zeta)(a^2-b^2)} \left[B (\frac{a^2+\xi}{b^2+\xi})^{1/2} (1 - \frac{\xi}{a^2+\xi}) - C (\frac{b^2+\xi}{a^2+\xi})^{1/2} (1 - \frac{\xi}{b^2+\xi}) \right]$$

$$\left. - \nu (B - C) \int_\xi^\infty \frac{s\,ds}{(a^2+s)(b^2+s)\sqrt{Q(s)}} \right), \quad = 0 \quad . \tag{60b}$$

where B and C are given by eq. (50). Near the crack boundary, eqs. (56) may be applied to simplify eqs. (60):

$$\frac{\tau_{xz}}{2\mu} = \frac{2\cos\phi}{ab}\sqrt{\frac{a}{b}} \left(\left[(\frac{a}{b})^2 B - C \right] (\frac{\nu b^2 \sin\phi}{a^2 \sin^2\phi + b^2 \cos^2\phi}) \right.$$

$$\left. - B \right) \frac{(a^2 \sin^2\phi + b^2 \cos^2\phi)^{1/4}}{\sqrt{2r}} + 0 \, (r^{1/2}) \tag{61a}$$

$$\frac{\tau_{yz}}{2\mu} = \frac{2\sin\phi}{ab}\sqrt{\frac{b}{a}} \left(\left[(\frac{b}{a})^2 C - B \right] (\frac{\nu a^2 \cos\phi}{a^2 \sin^2\phi + b^2 \cos^2\phi}) \right.$$

$$\left. - C \right) \frac{(a^2 \sin^2\phi + b^2 \cos^2\phi)^{1/4}}{\sqrt{2r}} + 0 \, (r^{1/2}) \tag{61b}$$

Before $k_j(j=2,3)$ can be found, the following transformations

$$\tau_{nz} = \tau_{xz} \cos\beta + \tau_{yz} \sin\beta \quad . \tag{62a}$$

$$\tau_{tz} = - \tau_{xz} \sin\beta + \tau_{yz} \cos\beta \quad . \tag{62b}$$

should be performed to obtain the normal and tangential components of the shear stress. In eqs. (62), β is the angle between the x-axis and the line

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normal to the crack border in the plane $z = 0$. It is not difficult to show that for a given value of ϕ, i.e., a fixed position on the crack boundary, β is known either from

$$\sin \beta = a \sin \phi \ (a^2 \sin^2\phi + b^2 \cos^2\phi)^{-1/2} \ .$$

or

$$\cos \beta = b \cos \phi \ (a^2 \sin^2\phi + b^2 \cos^2\phi)^{-1/2}$$

Putting eqs. (61) into (62), τ_{nz} and τ_{tz} may be written as

$$\tau_{nz} = \frac{k_2}{\sqrt{2r}} + 0 \ (r^{1/2}) \quad , \quad \tau_{tz} = \frac{k_3}{\sqrt{2r}} + 0 \ (r^{1/2}) \quad .$$

where

$$k_2 = - \ \frac{4\mu}{\sqrt{ab}} \ (a^2 \sin^2\phi + b^2 \cos^2\phi)^{-1/4} \ (B \cos^2\phi + C \sin^2\phi) \ . \tag{63a}$$

$$k_3 = \frac{4\mu \ (1-\nu)}{\sqrt{ab}} \ (a^2 \sin^2\phi + b^2 \cos^2\phi)^{-1/4} \ (\frac{a}{b} \ B - \frac{b}{a} \ C) \sin \phi \cos \phi \ . \tag{63b}$$

It is worthwhile to mention that in the skew-symmetrical problem of an elliptical crack, there exists a combination of "edge-sliding" and "tearing" movements of the crack surfaces as both k_2 and k_3 occur simultaneously. Hence, the shape to which the crack would grow will depend upon a function of k_2, k_3 and is no longer a priori evident as in the case of the opening mode. For $\nu = 1/3$, the values of k_2 and k_3 against ϕ are shown in curves by Figs. 2 and 3. Note that k_2 varies with ϕ in a manner similar to k_1 as in Fig. 1. Inspection of Fig. 3 reveals the interesting fact that the four points at which the crack border intersects the major and minor axes are under the action of "edge-sliding" type of displacement only, since k_3

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vanishes at those places. The maximum values of k_3 shift as the ellipticity is changed. This is clearly illustrated in Fig. 3.

In the degenerate case of decreasing ellipticity, i.e., as $b \to a$, eqs. (63) reduce to

$$k_2 = (\frac{1+\nu}{1-\nu}) \frac{2 \mu \alpha Q_0}{3\pi} a^{3/2} \quad , \quad k_3 = 0 \quad . \tag{64}$$

and the "tearing" mode disappears completely. This result is in agreement with that obtained by Florence and Goodier [5][18].

It should be emphasized that the angular distribution of the three-dimensional thermal stresses near the crack border is found to be the same as those in an elastic body undergoing deformation at constant temperatures [6]. The two-dimensional case was discussed by Sih in [12]. In retrospect, this justifies the application of the Griffith-Irwin theory of fracture to cracks owing to thermal disturbances.

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[18] The sum of the normal stresses, $(\sigma_r + \sigma_\phi)$ or $(\sigma_{xx} + \sigma_{yy})$ in [5], on the plane of the penny-shaped crack is incorrect. It can be seen from eqs. (47a), (47b) and (49) in the present paper that

$$(\frac{ab}{8\mu}) (\sigma_{xx} + \sigma_{yy})_{z=0} = - [(\frac{1+\nu}{1-\nu}) \frac{ab^2 \alpha Q_0}{2C(k)} + (1+\nu) (B+C)] (1 - \frac{x^2}{a^2} - \frac{y^2}{b^2})^{1/2}$$

$$+ (1+\nu) (B \frac{x^2}{a^2} + C \frac{y^2}{b^2}) (1 - \frac{x^2}{a^2} - \frac{y^2}{b^2})^{-1/2} \quad , \quad \xi = 0 \quad .$$

and hence, for $a = b$, the above equation becomes

$$(\frac{1-\nu}{1+\nu}) \frac{(\sigma_{xx} + \sigma_{yy})_{z=0}}{4 \mu \alpha Q_0 a} = - \frac{1}{\pi} [(1-\nu) \sqrt{1 - \rho^2} + \frac{1}{3} (1+\nu) \frac{1}{\sqrt{1 - \rho^2}}] \quad , \quad \rho < 1 \quad .$$

which is the revised form of eq. (28) in [5].

Remarks on a Class of Boundary Problems

A more general approach to the problem of finding the steady-state temperature field in ellipsoidal coordinates will be considered. Once the structure of T(x,y,z) is known, a number of thermoelastic problems of cracks may be solved by the method described earlier. For brevity sake, discussion will be restricted to the symmetrical problem where

$$T(x,y,z) = T(x,y, - z) \quad .$$

The formulation for the skew-symmetrical temperature problem follows in the same way.

Suppose that the surfaces of an elliptical crack is thermally disturbed with the following boundary conditions:

$$T = T(x,y) \quad , \xi = 0 \quad . \tag{65a}$$

$$\frac{\partial T}{\partial z} = 0 \quad , \eta = 0 \quad . \tag{65b}$$

which reduces to eqs. (24) for T(x,y,0) = constant. A well-known solution of eq. (1) in potential theory is [13]

$$T(x,y,z) = \int \int_{\Sigma} \frac{q(x',y')}{R} dx'dy' \quad , \tag{66}$$

where

$$R^{-1} = [(x-x')^2 + (y-y')^2 + z^2]^{-1/2}$$

and q(x', y', 0) is the density of a distribution of mass in the space (x', y', z'). The integral in eq. (66) is extended over the region occupied by the ellipse $x^2/a^2 + y^2/b^2 = 1$. The requirement is that T(x,y,z) must be sufficiently regular at infinity. This is satisfied as

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$$T \to R_0^{-1} \int \int_{\Sigma} q(x',y') \, dx'dy' \quad , \text{ when } R_0 \to \infty$$

where

$$R_0 = (x^2 + y^2 + z^2)^{1/2} \quad .$$

Now, eq. (66) may be inserted into the last of eq. (5) and the result can be integrated to admit the representation

$$\psi(x,y,z) = \frac{\alpha}{2} \left(\frac{1+\nu}{1-\nu}\right) \cdot \frac{1}{2\pi} \int \int_{\Sigma} q(x',y') \, \log (R + z) \, dx'dy' \quad . \tag{67}$$

Apart from the factor $\frac{\alpha}{2} \left(\frac{1+\nu}{1-\nu}\right)$, eq. (67) is Boussinesq's three-dimensional logarithmic potential for a disk in the shape of Σ, whose mass density is $q(x,y)$. While $\psi(x,y,z)$ has continuous derivatives vanishing at infinity, the function $\psi(x,y,z)$ itself is unbounded as $R \to \infty$, since

$$\psi \to [\log (R + z)] \cdot \int \int_{\Sigma} q(x,y) \, dxdy \quad , \text{ when } R \to \infty$$

The boundary condition, eq. (65a), can be satisfied by taking

$$T(x,y) = \frac{1}{2\pi} \int \int_{\Sigma} \frac{q(x',y') \, dx'dy'}{[\, (x-x')^2 + (y-y')^2 \,]^{1/2}} \quad . \tag{68}$$

Integral of this type is used in the contact problems of the theory of elasticity, in particular, the problem of a perfectly rigid elliptical punch on an elastic half-space [11].

In ellipsoidal coordinates, the temperature field may be taken as

$$T(x,y,z) = \int_{\xi}^{\infty} \phi(\omega) \, \frac{ds}{\sqrt{Q(s)}} \quad . \tag{69}$$

where

$$\omega(s) = 1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s} - \frac{z^2}{s} \quad .$$

and $\phi(\omega)$ is any twice differentiable function in the interval $(0,1)$ with finite one-sided derivatives at the boundary points of the interval. Note that eqs. (65) are satisfied by

$$\frac{\partial T}{\partial z} = -\frac{2}{ab} (\xi\eta\zeta)^{1/2} \int_\xi^\infty \phi'(\omega) \frac{ds}{S\sqrt{Q(s)}} - 2 \phi(0) \frac{\left[\, \eta\zeta \,(a^2+\xi)(b^2+\xi)\, \right]^{1/2}}{ab \,(\xi-\eta)(\xi-\zeta)} \quad . \quad (70a)$$

and

$$T(x,y) = \int_0^\infty \phi \,(1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s}) \,\frac{ds}{\sqrt{Q(s)}} \quad . \tag{70b}$$

It should be pointed out that if $\phi(\omega)$ is a polynomial of degree n in x^2 and y^2, then $T(x,y)$ will be another polynomial of the same degree in x^2 and y^2.

Alternatively, eq. (69) may also be used to satisfy the boundary conditions

$$\frac{\partial T}{\partial z} = \begin{cases} q(x,y) & , \,\xi = 0 \quad . & (71a) \\ \\ 0 & , \,\eta = 0 \quad . & (71b) \end{cases}$$

By taking $q(x,y)$ to be a function of the variable

$$Z = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \quad .$$

then eq. (71a) gives

$$\phi(\omega) = -\frac{ab}{2\pi} \int_0^\omega \frac{q(z)\,dz}{\sqrt{\omega-z}} \tag{72}$$

In particular, if

$$q(Z) = q_0 \,Z^n \quad . \tag{73}$$

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eq. (72) becomes

$$\phi(\omega) = - \frac{abq_0}{2\sqrt{\pi}} \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \cdot [\omega(s)]^{n+1/2} \quad .$$

The temperature distribution may also be computed as

$$T(x,y,z) = - \frac{abq_0}{2\sqrt{\pi}} \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \int_{\xi}^{\infty} [\omega(s)]^{n+1/2} \frac{ds}{\sqrt{Q(s)}} \tag{74}$$

For $n = p - 1/2$, where p is a positive integer, the integrand in eq. (74) can be expanded and the integration may be carried out term by term[19].

A few cases of eq. (74) are

(a) $n = - 1/2$, eq. (74) reduces to eq. (25) with

$$q_0 = \frac{T_0}{bK(k)} \quad .$$

(b)   The integral corresponding to $n = 0$ has yet to be evaluated.

(c)   The temperature $T(x,y,z)$ for $n = 1/2$ is

---

[19]In the special case of $a = b$, eq. (74) can be integrated for all values of n giving

$$T = \frac{bq_0}{\sqrt{\pi}} \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} (k')^{2(n+1)} I_{2(n+1)} , \xi = 0 \quad .$$

where

$$I_{2n} = \int_{0}^{K} \frac{dt}{(dn\ t)^{2n}} \quad .$$

$$T = -\frac{bq_0 K(k)}{2\ a^2\ k^2} \left(a^2k^2 - \left[\ 1 - \frac{E(k)}{K(k)}\ \right] x^2 - \left[\ \left(\frac{a}{b}\right)^2 \frac{E(k)}{K(k)} - 1\ \right] y^2 \right) , \quad \xi = 0 \ .$$

Conclusion

Three-dimensional problems of cracks in thermoelasticity have been formulated. The displacements and stresses are expressed in terms of potential functions, which are valid for planes of discontinuities bounded by smooth curves. Of fundamental interest is the problem of a flat elliptical crack whose surfaces are thermally disturbed. Exact solutions are obtained and reduced to the limiting cases discussed by previous authors [3-5].

Several essential features of the stability behavior of an elliptical crack, heated on its surfaces, are discussed in connection with the Griffith-Irwin theory of fracture. The concept of stress-intensity factor is introduced. In general, all three types of k-factors are present in structural members undergoing thermal changes. Therefore, a criterion of fracture will have to depend on a function of k_1, k_2, and k_3 to reach some critical value, say f_{cr} at which unstable crack extension would start. The function f_{cr} must be determined experimentally.

APPENDIX

(1) The following integrals are essential in the calculation of displacements and thermal stresses:

$$\int_{\xi}^{\infty} \frac{ds}{\sqrt{Q(s)}} = \frac{2}{a} \cdot u \ , \tag{75a}$$

$$\int_{\xi}^{\infty} \frac{ds}{(a^2+s)\sqrt{Q(s)}} = \frac{2}{a^3k^2} \left[\ u - E(u)\ \right] \ , \tag{75b}$$

$$\int_{\xi}^{\infty} \frac{ds}{(b^2+s)\sqrt{Q(s)}} = \frac{2}{a^3k^2} \left[\ \left(\frac{a}{b}\right)^2 E(u) - u - \left(\frac{a^2-b^2}{b^2}\right) \frac{sn\ u\ cn\ u}{dn\ u}\ \right] \ , \tag{75c}$$

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$$\int_{\xi}^{\infty} \frac{s \, ds}{(a^2+s)\sqrt{Q(s)}} = \frac{2}{ak^2} \left[E(u) - k'^2 u \right] \quad , \tag{75d}$$

$$\int_{\xi}^{\infty} \frac{s \, ds}{(b^2+s)\sqrt{Q(s)}} = \frac{2}{ak^2} \left[u - E(u) + k^2 \frac{sn \, u \, cn \, u}{dn \, u} \right] \quad , \tag{75e}$$

$$\int_{\xi}^{\infty} \frac{ds}{(a^2+s)(b^2+s)\sqrt{Q(s)}} = \frac{2}{a^5 k^4} \left[\frac{(a^2+b^2)}{b^2} E(u) - 2 u - \frac{(a^2-b^2)}{b^2} \frac{sn \, u \, cn \, u}{dn \, u} \right] , \tag{75f}$$

$$\int_{\xi}^{\infty} \frac{s \, ds}{(a^2+s)^2 \sqrt{Q(s)}} = \frac{2}{3 \, a^3 k^2} \left[(1 + \frac{k'^2}{k^2}) E(u) - 2 \frac{k'^2}{k^2} u - sn \, u \, cn \, u \right.$$
$$\left. \cdot dn \, u \right] \quad . \tag{75g}$$

$$\int_{\xi}^{\infty} \frac{s \, ds}{(a^2+s)(b^2+s)\sqrt{Q(s)}} = \frac{2}{a^3 k^2} \left[(\frac{a^2+b^2}{a^2-b^2}) u - \frac{2}{k^2} E(u) + \frac{sn \, u \, cn \, u}{dn \, u} \right] \quad . \tag{75h}$$

(2) For the skew-symmetrical part of the problem, it is necessary to find

$$\int_{z}^{\infty} z \cdot \Lambda (x,y,z) \, dz \quad , \qquad \int_{z}^{\infty} z \cdot \Omega (x,y,z) \, dz \quad .$$

where the functions $\Lambda (x,y,z)$ and $\Omega (x,y,z)$ are themselves integrals given by eq. (49). Hence, integral of the type

$$I = \frac{1}{2} \int_{z}^{\infty} z \left[\int_{\xi}^{\infty} (\frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{s} - 1) \frac{ds}{\sqrt{Q(s)}} \right] dz \quad . \tag{76}$$

is involved. Using formula given by eq. (38), eq. (76) becomes

$$I = -\frac{1}{8} \int_{\xi}^{\infty} (\frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{s} - 1)^2 \cdot \frac{s \, ds}{\sqrt{Q(s)}} \quad . \tag{77}$$

From eq. (77) follows then

$$\frac{\partial^2 I}{\partial x^2} = -\frac{1}{2} \int_{\xi}^{\infty} (\frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{s} - 1) \frac{s \, ds}{(a^2+s)\sqrt{Q(s)}} - x^2 \int_{\xi}^{\infty} \frac{s \, ds}{(a^2+s)^2 \sqrt{Q(s)}} \quad ,$$
$$\tag{78a}$$

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$$\frac{\partial^3 I}{\partial x^2 \partial y} = - y \int_\xi^\infty \frac{s \, ds}{(a^2+s)(b^2+s)\sqrt{Q(s)}} + \frac{2 \, x^2 y \, \xi^{3/2}}{(a^2+\xi)^{3/2}(b^2+\xi)^{1/2}(\xi-\eta)(\xi-\zeta)} \quad . \quad (78b)$$

$$\frac{\partial^3 I}{\partial x \partial y^2} = - x \int_\xi^\infty \frac{s \, ds}{(a2+s)(b2+s)\sqrt{Q(s)}} + \frac{2 \, xy^2 \, \xi^{3/2}}{(a2+\xi)^{1/2}(b2+\xi)^{3/2}(\xi-\eta)(\xi-\zeta)} \quad . \quad (78c)$$

Similar expressions for partial derivatives of I with respect to x,y,z may be obtained for the computation of stresses and displacements. It is now apparent that the solutions to the integrals in eqs. (78) can be obtained from eqs. (75).

References

1. Nowacki, W. (1962), "Thermoelasticity", PWN-Polish Scientific Publishers, Warsaw.

2. Boley, B. A. and Weiner, J. H. (1960), "Theory of Thermal Stresses", John Wiley and Sons, Inc., New York.

3. Olesiak, Z. and Sneddon, I. N. (1960), "The Distribution of Thermal Stress in an Infinite Elastic Solid Containing a Penny-Shaped Crack", Arch. Rational Mech. Anal., Vol. 4, p. 238.

4. Deutsch, E. (1965), "The Distribution of Axisymmetric Thermal Stress in an Infinite Elastic Medium Containing a Penny-Shaped Crack", Int. J. Engng. Sci. Vol. 3, p. 485.

5. Florence, A. L. and Goodier, J. N. (1963), "The Linear Thermoelastic Problem of Uniform Heat Flow Disturbed by a Penny-Shaped Crack", Int. J. Engng. Sci., Vol. 1, p. 533.

6. Kassir, M. K. and Sih, G. C. (1965), "Three-Dimensional Stress Distribution Around an Elliptical Crack under Arbitrary Loadings", to appear in the J. of Appl. Mech.

7. Irwin, G. R. (1960), "Fracture Mechanics", Structural Mechanics, Pergamon Press, New York, p. 557.

8. Green, A. E. and Sneddon, I. N. (1950), "The Distribution of Stress in the Neighborhood of a Flat Elliptical Crack in an Elastic Solid", Proc. Cambridge Phil. Soc., Vol. 46, p. 159.

9. Whittaker, E. T. and Watson, G. N. (1962), "Modern Analysis", Fourth ed., Cambridge University Press, London.

10. Lamb, H. (1924), "Hydrodynamics", Fifth ed., Cambridge University Press, London.

11. Galin, L. A. (1947), "On the Pressure of a Stamp of Elliptical Platform on an Elastic Half-Space", Prikl. matem. i. mekh., Vol. 11, No. 2 (in Russian).

12. Sih, G. C. (1962), "On the Singular Character of Thermal Stresses Near a Crack Tip", J. of Appl. Mech., Vol. 29, p. 587.

13. Kellogg, O. D. (1929), "Foundations of Potential Theory", Frederick Ungar Publishing Co., New York.

$(\dfrac{\pi\ k_1}{4\ \mu\ \alpha T_o b^{1/2}})$

1.50

$\dfrac{a}{b} = 10$

$\dfrac{a}{b} = 6$

$\dfrac{a}{b} = 2$

$\dfrac{a}{b} = 1$

1.00

0.50

$\nu = 1/3$

0.00

0   15°   30°   45°   60°   75°   90°

Angle ϕ
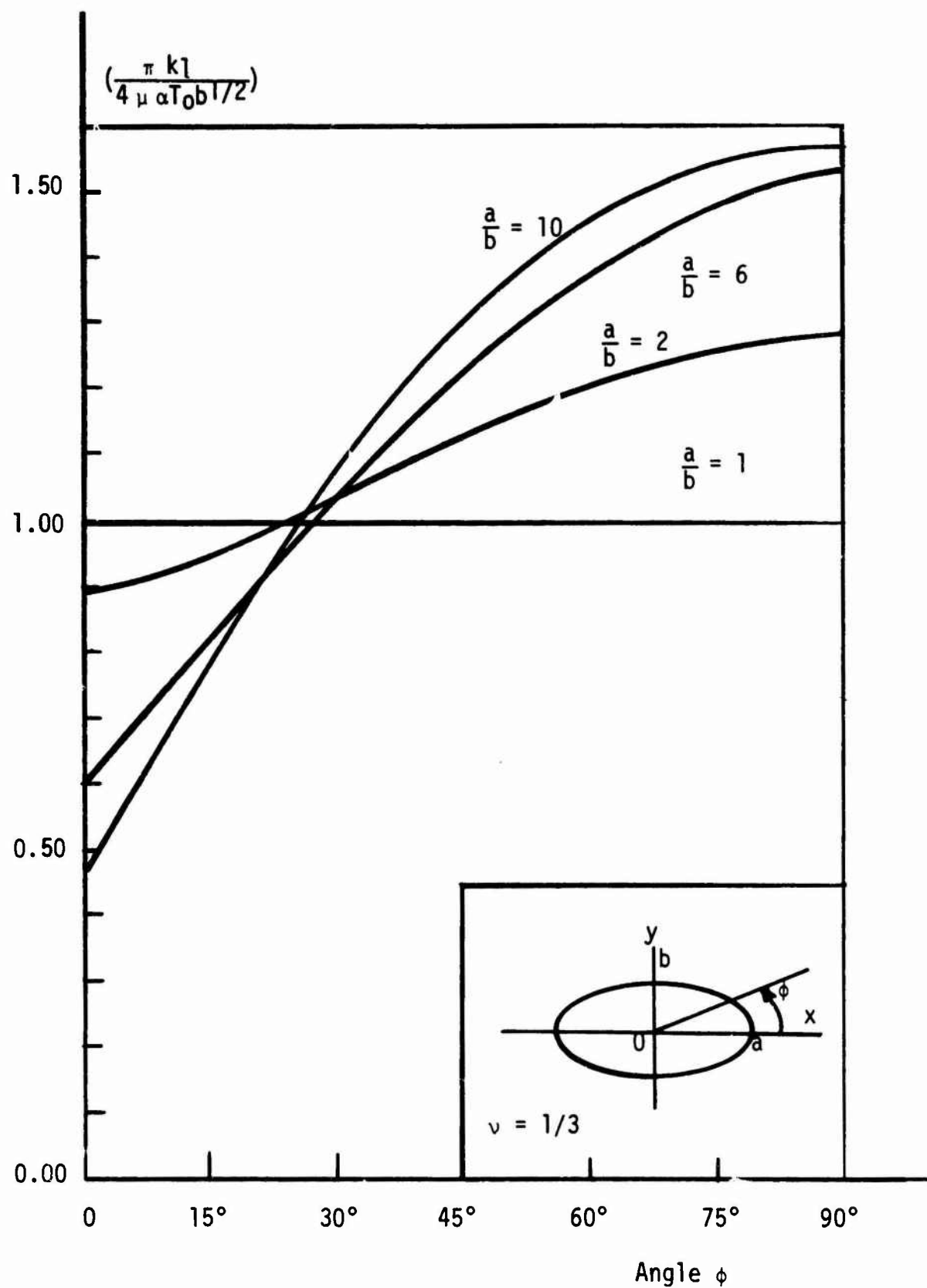
Figure (1) - "Opening" mode of fracture
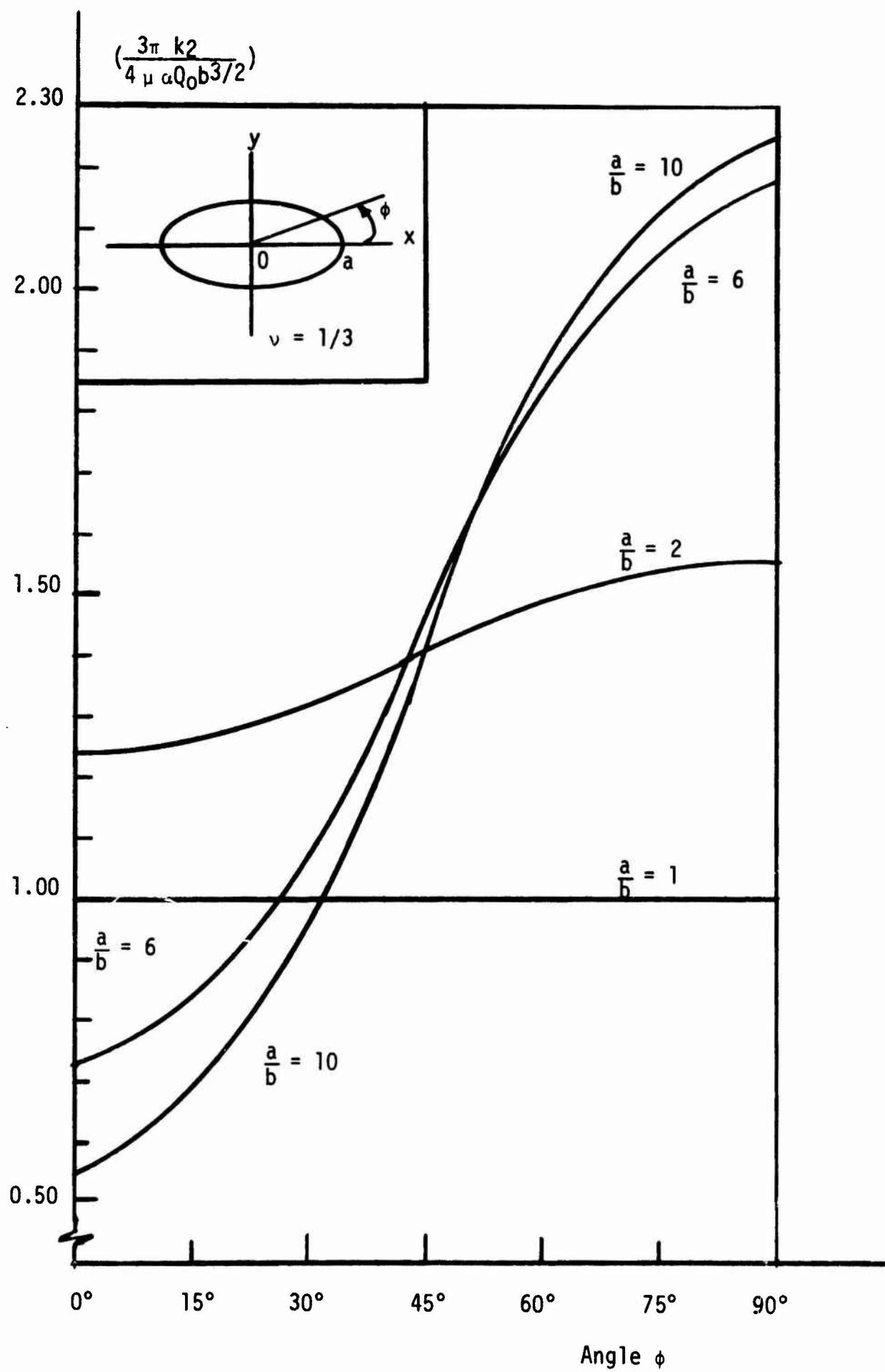
Figure (2): "Edge-sliding" mode of fracture
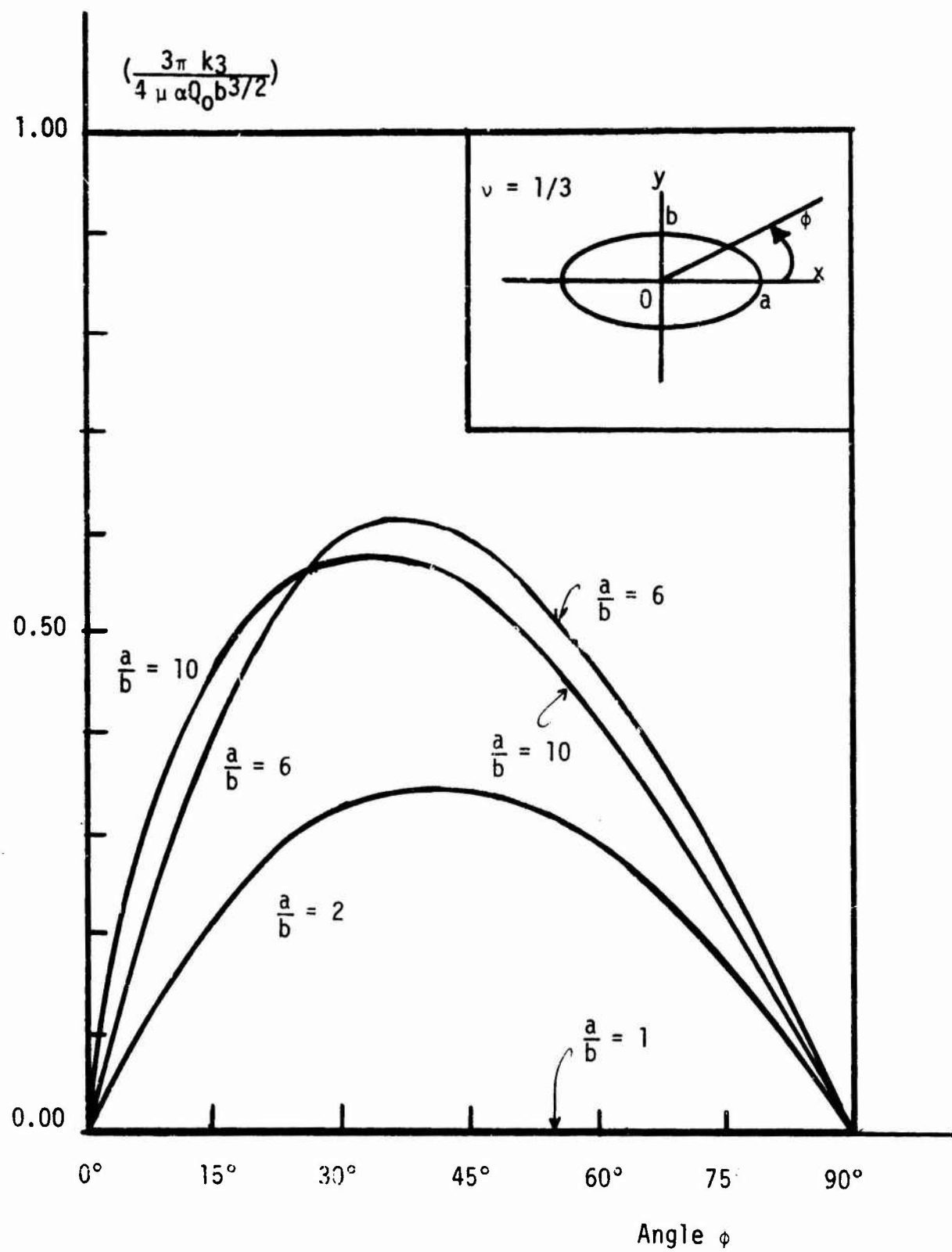
Figure (3): "Tearing" mode of fracture

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